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Hyper-symplectic structures on integrable systems

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Abstract

We prove that an integrable system over a symplectic manifold whose symplectic form is covariantly constant carries a natural hyper-symplectic structure. Moreover, a special Kähler structure is induced on the base manifold.

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1. Introduction

There exist deep relations between supersymmetric gauge theories and integrable systems, which were extensively investigated by Donagi and Witten [5] in their seminal paper. In particular, it turns out that a key notion in both frameworks is that of special Kähler manifold. According to the terminology fixed in [7], a special Kähler manifold is a Kähler (or more generally, pseudo-Kähler) manifold equipped with a flat torsion-free connection ∇ such that the covariant derivative of the Kähler form and the covariant differential of the complex structure are both equal to zero (for details see Section 3). The cotangent bundle of any special Kähler manifold carries a compatible hyper-symplectic structure; moreover, one can induce on it, by means of the connection ∇ , a natural structure of algebraically completely integrable system [7].

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In this note, we want to take an alternative point of view, looking at things the other way round. Indeed, our starting point is a classical integrable system *à la* Liouville-Arnol’d, i.e. a fibration $X \rightarrow B$, where X is a symplectic manifold and the fibres are Lagrangian tori. On such a fibration a flat torsion-free connection ∇ is naturally defined. Assuming the existence of a covariantly constant symplectic structure Ω on B , we prove the following:

- X admits a compatible hyper-symplectic structure (Theorem 2.2);
- B is a special Kähler manifold (Corollary 3.3).

Our approach should be compared to the treatment of special Kähler geometry developed by Hitchin to describe the geometry of the moduli space M of deformations of a complex Lagrangian submanifold of a complex Kählerian manifold [8]. In particular, our construction implies the existence of a bi-Lagrangian immersion of the base manifold B into the total space X (see Remark 3.4).

2. The basic construction

Let (X, ω) be a connected symplectic manifold of dimension $2N$, together with a projection $\pi : X \rightarrow B$ whose fibres $F_b = \pi^{-1}(b)$ are compact connected Lagrangian submanifolds of X . We shall call $\pi : X \rightarrow B$ an integrable system. By the Liouville-Arnol’d theorem [2], the fibres F_b are isomorphic to N -dimensional tori. The homology groups $H_1(F_b, \mathbb{Z})$ define a sheaf \mathcal{L} on the base manifold B , called the period lattice of the fibration $\pi : X \rightarrow B$ [6]. The dual sheaf of \mathcal{L} , $\text{Hom}(\mathcal{L}, \mathbb{Z})$ is isomorphic to the sheaf $R^1\pi_*\mathbb{Z}$. The sheaf of \mathcal{C}_B -modules

$$\mathcal{D} := \mathcal{L} \otimes_{\mathbb{Z}} \mathcal{C}_B^{\infty}$$

is locally free; \mathcal{L}_b is a lattice in the fibre D_b of the associated vector bundle D over B , and the quotient D_b/\mathcal{L}_b is precisely the Liouville-Arnol’d torus at the point b . We have the natural isomorphism $\pi^*D \simeq \text{Vert}(TX)$. We can define (locally) action-angle canonical coordinates $(I_1, \dots, I_N, \varphi_1, \dots, \varphi_N)$: the vertical Hamiltonian vector fields X_i associated to the actions I_i single out a basis $\{\gamma_i\}$ of local sections of \mathcal{L} and the angle variables $\{\varphi_i\}$ on the fibre F_b satisfy the relation

$$\frac{1}{2\pi} \int_{\gamma_i} d\varphi_k = \delta_k^i.$$

The coordinates I_i can be thought as affine coordinates on the base manifold B ; by associating dI_i to X_i , we obtain the natural isomorphism

$$T^*B \simeq D \tag{1}$$

or equivalently $TB \simeq R^1\pi_*\mathbb{R} \otimes_{\mathbb{R}} \mathcal{C}_B^{\infty}$. The Gauß-Manin connection of the local system $R^1\pi_*\mathbb{R}$ induces a torsion-free flat connection ∇ on TB ; clearly, the vector fields $\partial/\partial I_i$ are parallel w.r.t. ∇ . The subsheaf $\mathcal{L} \subset \mathcal{D}$ defines, via the identification (1), a Lagrangian covering of B , that will be denoted by $A \subset T^*B$; the monodromy of A coincides with the holonomy of ∇ . According to [6], if the fibration $\pi : X \rightarrow B$ admits a section, then there

is an isomorphism $X \simeq T^*B/A$ (which is an isomorphism of symplectic manifolds fibred in Lagrangian tori if the section is Lagrangian).

One has the identification $\text{Vert}(TX) \simeq \pi^*T^*B$. Let us consider the dual fibration $\hat{X} = R^1\pi_*\mathbb{R}/R^1\pi_*\mathbb{Z}$, naturally endowed with a projection $\hat{\pi} : \hat{X} \rightarrow B$. The identification $\text{Vert}(T\hat{X}) \simeq \hat{\pi}^*TB$ can be plugged in the Atiyah sequence

$$0 \rightarrow \text{Vert}(T\hat{X}) \rightarrow T\hat{X} \rightarrow \hat{\pi}^*TB \rightarrow 0 \tag{2}$$

which is split by the Gauß-Manin connection. In this way, we get a decomposition

$$T\hat{X} = \hat{\pi}^*T^*B \oplus \hat{\pi}^*TB. \tag{3}$$

According to [1,3], an almost-complex structure J is defined on \hat{X} by setting $J(U, V) = (-V, U)$; actually, J is integrable because the Gauß-Manin connection is flat. If we introduce on \hat{X} local coordinates $(I_1, \dots, I_N, \Phi_1, \dots, \Phi_N)$, where the Φ 's are dual coordinates to the φ 's, then the holomorphic coordinates are given by $z_j = I_j + i\Phi_j$. If the fibration $\pi : X \rightarrow B$ has a section, then X can be identified to \hat{X} , in such a way that the natural lifting of the Gauß-Manin connection coincides with the connection we have introduced on \hat{X} and the Φ 's coincide with the angles φ 's.

Let us suppose that the base manifold B has a symplectic structure Ω ; this implies $N = 2n$. We shall be concerned only with local properties, so that we can assume that a Lagrangian section $\sigma_\omega : B \rightarrow X$ does exist.

If Ω is covariantly constant, i.e. $\nabla\Omega = 0$, then, the splitting $TX = \pi^*T^*B \oplus \pi^*T^*B$ (together with the isomorphism $TB \simeq T^*B$ provided by Ω itself) can be used to construct a new symplectic form $\chi = -\Omega \oplus \Omega$ on X . The condition $\nabla\Omega = 0$ is equivalent to the fact that the action coordinates I_i can be assumed to be canonical w.r.t. Ω ; dropping the indices, we shall write short $I = (x, y)$ and $\Omega = dx \wedge dy$. By using the same notations, we also have $\varphi = (p, q)$ and $\omega = d\varphi \wedge dI = dp \wedge dx + dq \wedge dy$. Finally, we get

$$\chi = -dp \wedge dq + dx \wedge dy.$$

We shall denote by J_ω and J_χ the complex structures associated, respectively, to the symplectic structures ω and χ on the dual fibration \hat{X} (in principle, the complex structure J_χ is defined only locally). In the local coordinates (x, y, P, Q) on \hat{X} , one has

$$J_\omega = dx \otimes \frac{\partial}{\partial P} - dP \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial y},$$

$$J_\chi = -dP \otimes \frac{\partial}{\partial Q} + dQ \otimes \frac{\partial}{\partial P} + dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}.$$

Let us define $K = J_\omega \circ J_\chi$.

Lemma 2.1. *K is a complex structure.*

Proof. An easy computation yields:

$$K(dx) = dQ, \quad K(dy) = -dP, \quad K(dQ) = -dx, \quad K(dP) = dy.$$

Therefore, $K \circ K = -\text{Id}$. □

In local coordinates, we can write:

$$K = -dx \otimes \frac{\partial}{\partial Q} + dQ \otimes \frac{\partial}{\partial x} - dP \otimes \frac{\partial}{\partial y} + dy \otimes \frac{\partial}{\partial P}.$$

The holomorphic coordinates $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ induced by K satisfy the relations $d\alpha = dx + i dQ$, $d\beta = dP + i dy$. On X we define the symplectic form

$$\sigma = dq \wedge dx + dy \wedge dp.$$

We shall write J_σ instead of K . Summing up, we have proved the following result.

Theorem 2.2. *Let $X \rightarrow B$ be an integrable system over a symplectic manifold (B, Ω) such that $\nabla\Omega = 0$. Then there exist (locally) a hyper-symplectic structure on X and a hyper-complex structure on the dual fibration \hat{X} .*

Proof. Lemma 2.1 shows that J_ω, J_χ and J_σ determine a hyper-complex structure on \hat{X} . It follows that $(\omega^{-1}\chi)^2 = -\text{Id}_{TX}$; hence, ω, χ and σ define a hyper-symplectic structure on X [9]. □

We can identify X and \hat{X} in such a way that the coordinates (x, y, p, q) correspond to the coordinates (x, y, P, Q) . As a straightforward consequence of Theorem 2.2 we obtain the following corollary.

Corollary 2.3. *Let $\rho_\omega : B \rightarrow X$ be a Lagrangian immersion w.r.t. the symplectic structure ω . Then the image $\rho_\omega(B)$ is a complex submanifold w.r.t. the complex structure J_χ .*

Since the holomorphic coordinates on X induced by J_χ are $(u, v) = (q + ip, x + iy)$, we can consider also $q + ip$ as holomorphic coordinates on B (to be precise, we need to identify TB and $\text{Vert}(TX)$ by means of the symplectic form ω). Given a Lagrangian immersion ρ_σ (resp., ρ_χ) w.r.t. σ (resp., χ), the analogue of Corollary 2.3 is readily stated: $\rho_\sigma(B)$ is a complex submanifold w.r.t. J_ω (resp., $\rho_\sigma(B)$ is a complex submanifold w.r.t. J_ω).

3. Special Kähler geometry on B

In this section we show how to recover from the data described in the previous sections a special Kähler structure on B [7,8]. We start with the following basic definition.

Definition 3.1 (Alekseevsky et al. [4]). A complex manifold (B, I) is called *special complex* if there is a flat, torsion-free (linear) connection ∇ such that:

$$d\nabla I = 0.$$

If there is a covariantly constant symplectic form Ω on B (i.e. $\nabla\Omega = 0$), then the triple (M, I, Ω) is called *special symplectic*.

Let us suppose that a Lagrangian section $\rho_\sigma : B \rightarrow X$ does exist. From Corollary 2.3, we have that $\rho_\sigma(B)$ is a complex submanifold w.r.t. the complex structure J_ω . Moreover, the holomorphic coordinates induced by this complex structure on X are $(z, w) = (x + ip, y + iq)$. These coordinates define, via ρ_σ , a double set of holomorphic coordinates on B , $z = x + ip$ and $w = y + iq$, while the restriction to $\rho_\sigma(B)$ of the complex structure J_ω induces a complex structure on B that can be written in local coordinates as:

$$I = - \left(dp \otimes \frac{\partial}{\partial x} + dq \otimes \frac{\partial}{\partial y} \right). \quad (4)$$

Since the coordinates (x, y) are flat w.r.t. the Gauß-Manin connection we have:

$$d_\nabla I = 0.$$

So have proven the following result.

Proposition 3.2. *Under the same assumptions of Theorem 2.2 and assuming the existence of a Lagrangian section as above, the manifold B is special symplectic.*

Let us recall that a special symplectic manifold (B, Ω, I) is said to be *special Kähler* if the symplectic form Ω is I -invariant [4,7,8]. Notice that the 2-tensor $g(\cdot, \cdot) = \Omega(\cdot, I\cdot)$, whilst is always symmetric, in general is not positive definite. So, the name of special pseudo-Kähler would be more appropriate.

The symplectic form Ω can be written as $\Omega = dx \wedge dy$ (recall that the (x, y) 's are the flat symplectic coordinates on B) and a straightforward calculation shows that such form is invariant w.r.t. the complex structure defined in (4). Thus, under the same assumption as in Proposition 3.2, the following result is easily proved.

Corollary 3.3. *The manifold B is special Kähler, with (pseudo)-Kähler metric g given by $g = \Omega \circ I$.*

Remark 3.4. In [8, Theorem 2.4], Hitchin characterises special (pseudo)-Kähler manifolds as manifolds that can be (locally) identified as bi-Lagrangian submanifolds of $V \times V$, where V is a symplectic vector space. In our description of the special geometry defined on the base of the fibration, we use the existence of only one Lagrangian section ρ_σ even if B can be (clearly) identified via the Gauß-Manin connection with a ω -Lagrangian submanifold of the total space X .

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